Equations of motion

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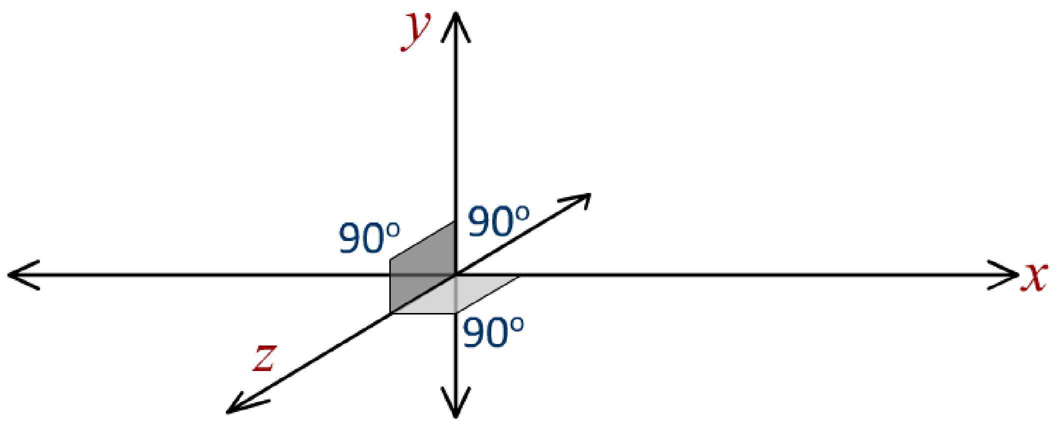
Note : The notes given in this file is no substitute to the much detailed discussion held in the online/contact classes with active participation of students. It , at best, serves the purpose of ready reference for important concepts/derivations covered in the classes.

Position

Position of a body refers to its location w.r.t. to reference.

A reference frame is a combination of mutually perpendicular axes intersecting at a common point called origin and a clock for keeping time.

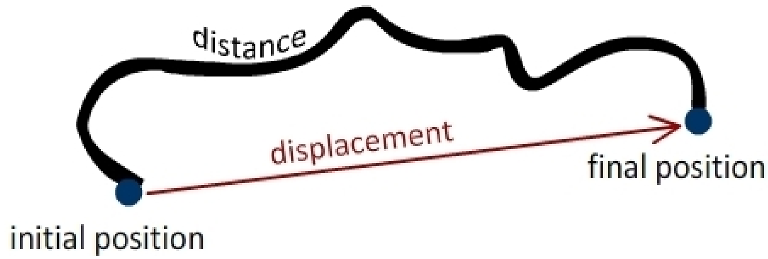


Number of coordinates required to uniquely locate the body depends on the kind of motion.

Motion	No. of coordinates	Examples
One dimensional	One (x)	Freely falling body
Two dimensional	Two (x, y)	Projectile
Three dimensional	Three (x, y, z)	A bird in flight

Distance

- It is the length of the path followed by body as it moves from its initial to its final position.
- It can be either zero (for a stationary body) or non-zero for a body in motion
- It is a scalar quantity
- SI unit is m
- It is measured by the odometer in our vehicles



Displacement

- It is given by the length of the shortest line drawn from the initial position to the final position a body undergoing motion.
- It may be zero, positive or negative depending on the motion of the body
- It is a vector quantity (from initial to final position)
- SI unit is m
- It is calculated based on our observation.

- **Velocity** : It is defined as the rate of change of displacement w.r.t. time
- It is vector quantity
- SI unit of velocity is ms^{-1}
- Average velocity is given by displacement of the body in the given time interval divided by time.

$$v_{\text{avg}} = \frac{\Delta S}{\Delta t}$$

- Instantaneous velocity is given by the rate of displacement w.r.t. time

$$v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

- **Acceleration** : It is defined as the rate of change of velocity w.r.t. time
- It is vector quantity
- SI unit of acceleration is ms^{-2}
- Average acceleration is given by total change in velocity divided by the time interval

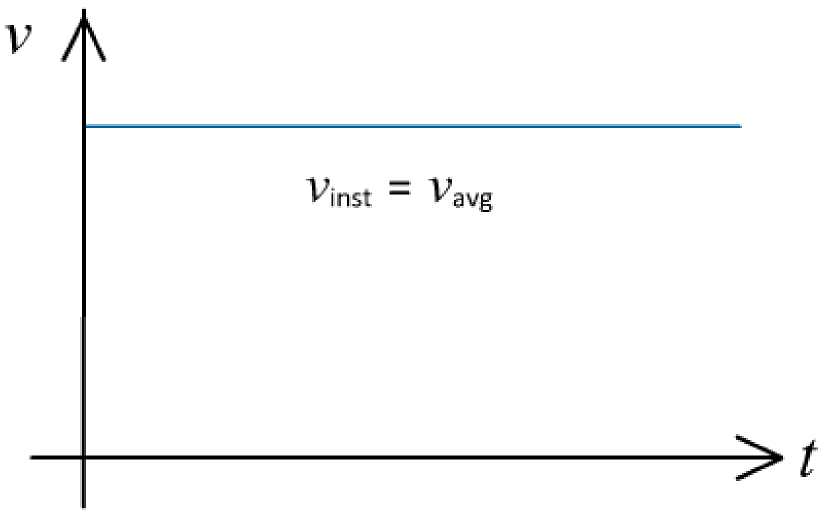
$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

- Instantaneous acceleration is given by the rate of change of velocity w.r.t. time

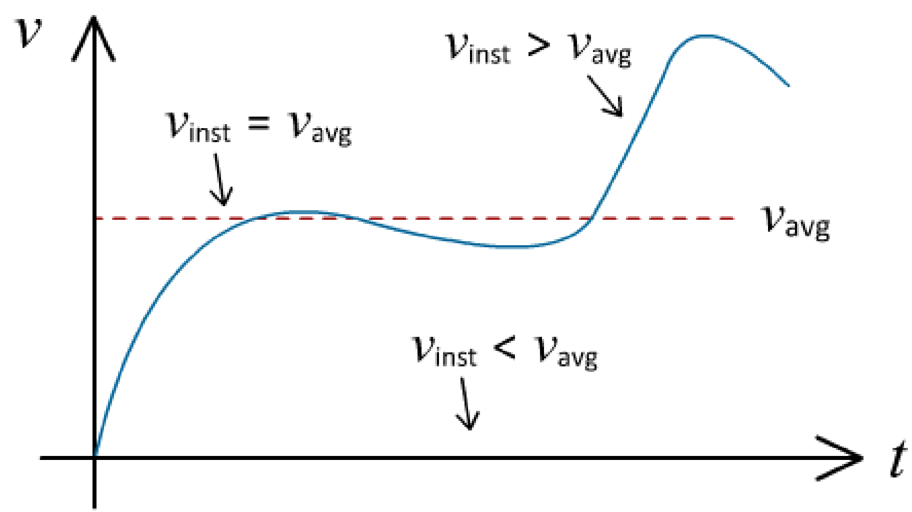
$$a_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

Average velocity is different from instantaneous velocity.

Average velocity is a measure of the overall rate change of displacement where as instantaneous velocity is a measure of the rate of change of displacement at a particular instant of time.



For a uniform motion average velocity is equal to instantaneous velocity for the complete duration of time.



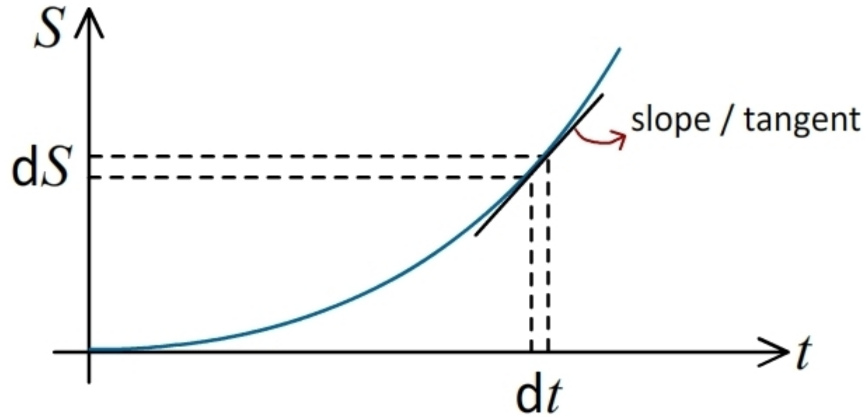
For a non-uniform motion average velocity may be equal to instantaneous velocity at one or more instants of time.

Body moving with uniform acceleration

Consider a body moving with uniform acceleration. Displacement of the body, as a function of time is given by the relation

$$S = ut + \frac{1}{2}at^2$$

Displacement-time graph of the body is as given below.



Click here for simulation

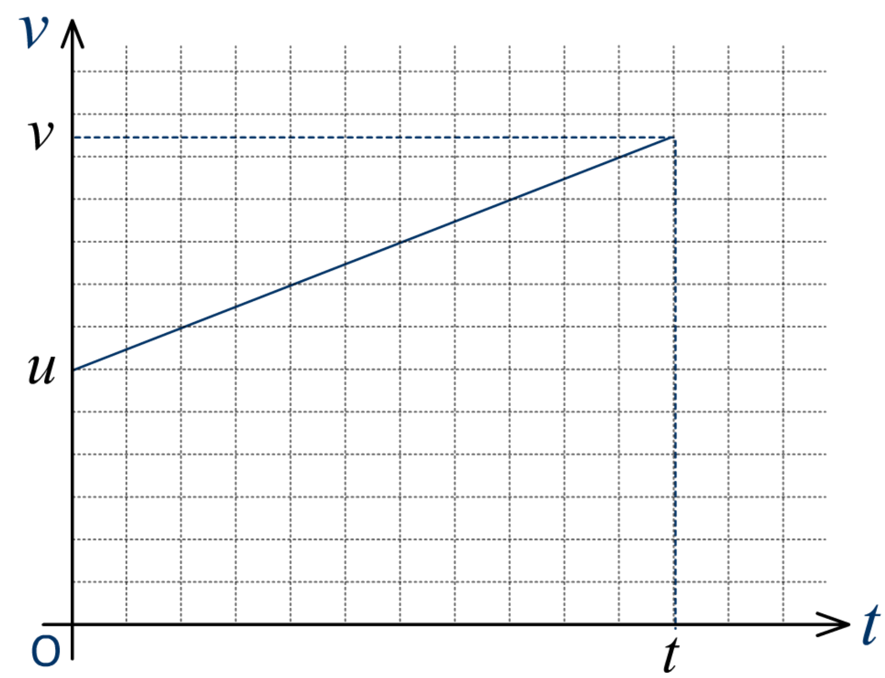
Instantaneous velocity of the body is given by the slope of the tangent drawn at a point (on the graph) at which the velocity is to be determined.

- Slope of $S-t$ plot gives the rate at which displacement occurs.
- For a certain interval of time, the slope gives average velocity (and in the process the finer details of any increase/decrease are lost)
- Choosing a very small interval of time gives the instantaneous velocity.
- In choosing a very small interval, slope of graph at that point becomes the tangent to the curve.

- Slope of $v-t$ plot gives the rate at which the velocity changes
- For a certain interval of time, the slope gives average acceleration (and in the process the finer details of any increase/decrease are lost)
- Choosing a very small interval of time gives the instantaneous acceleration.
- In choosing a very small interval, slope of graph at that point becomes the tangent to the curve.

- Area of acceleration versus time plot gives velocity
- Area of velocity versus time plot gives displacement

Equations of motion (from $v - t$ graph)



Consider a body having initial velocity u and moving uniform acceleration a to time t.

Velocity versus time plot for such a motion is as shown in the figure.

Slope of the plot is given by

$$\text{slope} = \frac{\Delta y}{\Delta x}$$

$$\text{slope} = \frac{\Delta v}{\Delta t}$$

$$\text{slope} = \frac{v - u}{t - 0}$$

$$\text{slope} = \frac{v - u}{t}$$

The rate of change of velocity is called acceleration, therefore

$$a = \frac{v - u}{t}$$

$$v = u + at \quad (1)$$

The above equation gives velocity as a function of time

Equations of motion (from $v-t$ graph)

Area under the velocity-time plot gives the displacement of the body.

Area under the curve of is given by

$$\text{Area} = A_1 + A_2 \quad \text{(i)}$$

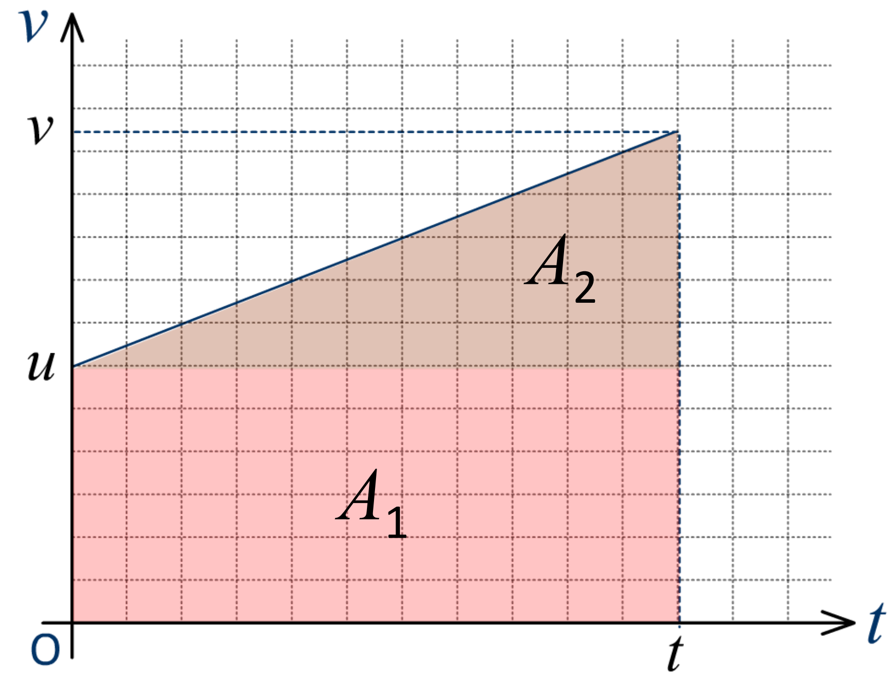
$$A_1 = u \times t \quad \text{(ii)}$$

$$A_2 = \frac{1}{2}\,\text{base} \times \text{alt}$$

$$A_2 = \frac{1}{2}\,t \times (v - u) \quad \text{(iii)}$$

Substituting (iii) and (ii) in (i) we get

$$S = ut + \frac{1}{2}\,t \times (v - u) \quad \text{(iv)}$$



Using the relation

$$v = u + at \quad \Rightarrow v - u = at$$

$$S = ut + \frac{1}{2}\,t \times at$$

$$S = ut + \frac{1}{2}at^2 \quad (2)$$

Equations of motion (from $v-t$ graph)

Displacement as a function of time is given by the relation

$$S = ut + \frac{1}{2}at^2 \quad \text{(i)}$$

Velocity as a function of time is given by the relation

$$v = u + at$$

$$t = \frac{v-u}{a} \quad \text{(ii)}$$

Substituting this in equation (i) we get

$$S = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

$$S = \left(\frac{v-u}{a}\right)\left(u + \frac{1}{2}a\left(\frac{v-u}{a}\right)\right)$$

$$S = \left(\frac{v-u}{a}\right)\left(u + \frac{v-u}{2}\right)$$

$$S = \left(\frac{v-u}{a}\right)\left(\frac{v+u}{2}\right)$$

$$v^2 - u^2 = 2aS \quad \text{(3)}$$

Equations of motion (using calculus)

Acceleration is defined as the rate of change of velocity w.r.t. time. Therefore

$$a = \frac{dv}{dt} \quad \text{(i)}$$

$$dv = a\,dt$$

$$\int_u^v dv = \int_0^t a\,dt$$

For constant acceleration we get

$$\int_u^v dv = a\int_0^t dt$$

$$v - u = at$$

$$v = u + at$$

Velocity is defined as the rate of change of displacement w.r.t. time. Therefore

$$v = \frac{dS}{dt} \quad \text{(i)}$$

$$dS = v\,dt$$

For constant acceleration we get

$$\int_i^f dS = \int_0^t (u + at)\,dt$$

$$S - S_o = ut + \frac{1}{2}at^2$$

$$S = S_o + ut + \frac{1}{2}at^2$$

S_o is the initial displacement of the body from the origin

Equations of motion (using calculus)

Acceleration is defined as the rate of change of velocity w.r.t. time. Therefore

$$a = \frac{dv}{dt} \quad \text{(i)}$$

$$a = \frac{dv}{dS}\frac{dS}{dt}$$

Using $v = \frac{dS}{dt}$ we get

$$a = \frac{dv}{dS} v$$

$$a\,dS = v\,dv$$

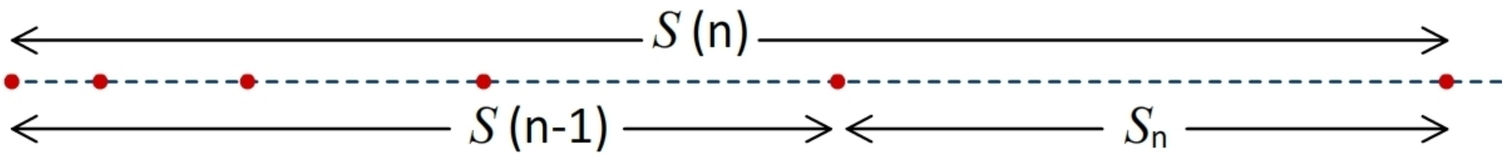
Integrating the above equation, for constant acceleration, we get

$$a\int_0^S dS = \int_u^v v\,dv$$

$$a\,S = \frac{v^2 - u^2}{2}$$

$$v^2 - u^2 = 2a\,S$$

Displacement in the n^{th} second of motion



Consider a body having an initial velocity (u) and moving with uniform acceleration (a) . Its displacement, as a function of time, is given by the relation

$$S = ut + \frac{1}{2}at^2$$

Displacement in the n^{th} second is

$$S(n) = un + \frac{1}{2}an^2$$

Displacement in the (n -1) th second is

$$S(n-1) = u(n-1) + \frac{1}{2}a(n-1)^2$$

Subtracting (ii) from (i) we get

$$S_n = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$

$$S_n = un + \frac{1}{2}an^2 - un + u - \frac{an^2}{2} + an - \frac{a}{2}$$

$$S_n = u + a\left(n - \frac{1}{2}\right) \quad (4)$$

Galileo's law of odd numbers

Distances covered by a free falling body in successive, equal intervals of time are in the ratio of the odd numbers (i.e. 1 : 3 : 5 : 7 : 9 . . .).

Displacement of a body in the n^{th} second of its motion is given by the relation

$$S_n = u + a\left(n - \frac{1}{2}\right)$$

In case of a freely falling body u = 0 and a = g. Therefore

$$S_n = g\left(n - \frac{1}{2}\right)$$

Displacement in the 1^{st} second is

$$S_1 = g\left(1 - \frac{1}{2}\right) \Rightarrow S_1 = g\left(\frac{1}{2}\right) \quad \text{(i)}$$

Displacement in the 2^{nd} second is

$$S_2 = g\left(2 - \frac{1}{2}\right) \Rightarrow S_2 = g\left(\frac{3}{2}\right) \quad \text{(ii)}$$

Displacement in the 3^{rd} second is

$$S_3 = g\left(3 - \frac{1}{2}\right) \Rightarrow S_3 = g\left(\frac{5}{2}\right) \quad \text{(iii)}$$

From equations (i), (ii) and (iii) the ratio is

$$S_1 : S_2 : S_3 = 1 : 3 : 5 \ldots$$

Does v = 0 necessarily imply that a = 0 ?

Acceleration of a body is the rate of change in its velocity as a function of time.

In determination of acceleration, a certain interval of time (dt) and the change in velocity (dv) is measured for that small interval of time. Therefore, acceleration is based on change in velocity and not velocity itself.

$$a = \frac{dv}{dt}$$

$$a = \frac{v_f - v_i}{t_f - t_i}$$

Acceleration is zero only when $v_f = v_i$ (both zero or both non-zero)

Acceleration may be non-zero when $v_f \neq 0$ or $v_i \neq 0$

Average velocity (in case of constant acceleration)

Average velocity of a body is given by

$$v_{\text{avg}} = \frac{S_{\text{total}}}{t_{\text{total}}}$$

Total displacement of the body is given by

$$S = ut + \frac{1}{2}at^2$$

$$v_{\text{avg}} = \frac{ut + \frac{1}{2}at^2}{t}$$

$$v_{\text{avg}} = u + \frac{1}{2}at$$

Using the relation for velocity

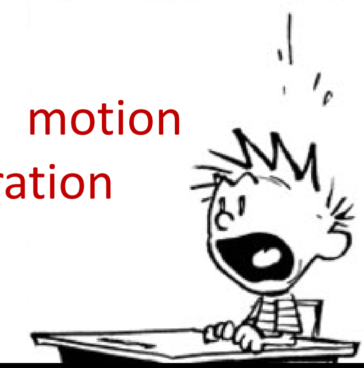
$$v = u + at$$

$$\Rightarrow at = v - u$$

$$v_{\text{avg}} = u + \frac{1}{2}(v - u)$$

$$v_{\text{avg}} = \frac{v + u}{2} \quad (5)$$

* Applicable only for motion with constant acceleration



Rest and motion are relative

The inference of motion of a body is always relative to the observer. An object that appears to be stationary for one observe may appear to be in motion for another observer.

Example Consider three persons A, B and C. A and B are in a train moving with a uniform speed. C is on the ground.
When A is observed by B, he infers that A is at rest w.r.t. him.
When C observes A she infers that A is moving (at the speed of the train) w.r.t to her (as she is on the ground)

Hobbes is at rest w.r.t. Calvin !
A person on ground would say that they are moving downhill !

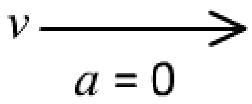
Velocity and acceleration : directions & magnitudes

Velocity is zero but acceleration is not zero :

Consider a body projected vertically up. At the highest point of its trajectory, its velocity s zero momentarily but the acceleration (due to gravity) remains constant.

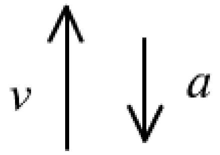
Velocity is not zero but acceleration is zero :

Uniform motion in a straight line (like the cruising speed of an aircraft)



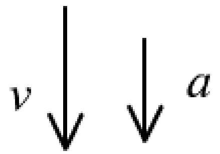
Velocity and acceleration in opposite directions :

During accent of a body projected vertically up, velocity is upwards but acceleration (due to gravity) is downwards



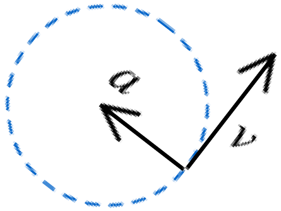
Velocity and acceleration are in same direction :

Velocity and acceleration of a freely falling body (or a during the descent of a body projected vertically up)



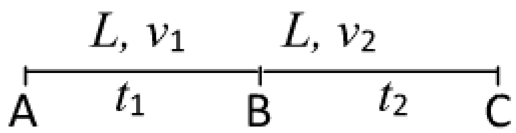
Velocity and acceleration are in different directions :

For a body in a uniform circular motion velocity is tangential and acceleration is radial.



Average velocity (some cases)

A vehicle travels half the distance L with speed v_1 and the other half with speed v_2. What is the average speed?



Average speed is given by

$$\overline{v}_{avg} = \frac{\overline{S}_{total}}{t_{total}} \qquad \text{(i)}$$

Time taken for each part of the motion is given by

$$t_1 = \frac{L}{v_1} \qquad t_2 = \frac{L}{v_2} \qquad \text{(ii)}$$

Substituting t_1 and t_2 in eq (i) we get

$$v_{avg} = \frac{2L}{\frac{L}{v_1} + \frac{L}{v_2}}$$

$$v_{avg} = \frac{2v_1 v_2}{v_1 + v_2}$$

Directions of displacement, velocity and acceleration

A lift coming down is just about to reach the ground floor. Taking the ground floor as origin and positive direction upwards for all quantities, which one of the following is correct?

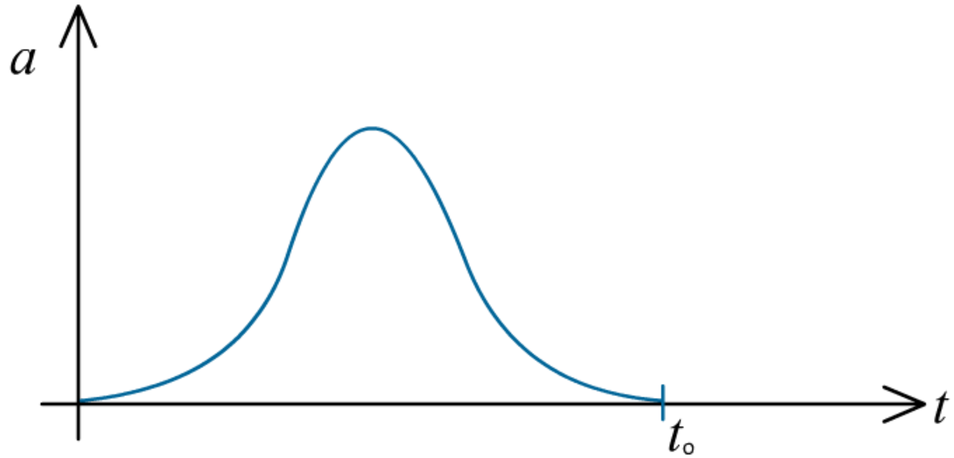
a) $x < 0, v < 0, a > 0$
b) $x > 0, v < 0, a < 0$
c) $x > 0, v < 0, a > 0$
d) $x > 0, v > 0, a > 0$

The body is above the ground hence its position coordinate $x > 0$
(Note: If x is considered as displacement then x is downwards and therefore $x < 0$)
Velocity is downwards therefore $v < 0$
Acceleration is upwards therefore $a > 0$.
(as the lift is slowing down it implies that acceleration in a direction opposite to the direction of instantaneous velocity)

Acceleration and changing direction of motion

A uniformly moving cricket ball is hit with a bat for a very short time and is turned back. Show the variation of its acceleration with time taking the acceleration in the backward direction as positive.

Assume that the ball comes in contact with the bat at t = 0 and remains in contact for a small interval of time t_o. During this interval of time acceleration of the ball is in the left hand side direction.



Motion with non-uniform acceleration

Can the equations of kinematics be used when acceleration varies with time?

No, the following equations of kinematics

(i.e. $v = u + at$ $S = ut + \frac{1}{2}at^2$ $v^2 - u^2 = 2aS$)

<u>cannot</u> be used when acceleration of the body is not constant

The general equations of motion that can be used for both constant and variable accelerations are

$$\overline{v} = \frac{\mathrm{d}\overline{S}}{\mathrm{d}t} \Rightarrow S = \int v \, \mathrm{d}t \qquad \overline{a} = \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} \Rightarrow \overline{v} = \int a \, \mathrm{d}t$$

The choice of limits for integration and computation of the integral depends on the nature of problem to be solved.

Terminal velocity

An object falling through fluid is observed to have an acceleration given by $a = g - bv$ where g is gravitational acceleration and b is a constant. After a long time is observed to fall with constant velocity. What would be the value of this constant velocity?

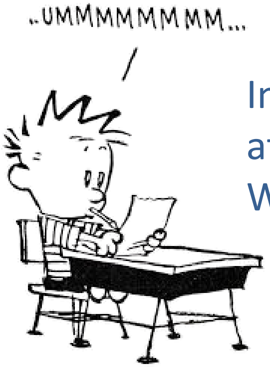
Given $a = g - bv$

When acceleration becomes zero we get

$$0 = g - bv$$

$$bv = g$$

$$v = g/b$$



In accelerated motion , under certain situations, the velocity of the body becomes constant after reaching a certain limit . This upper limit of velocity is called terminal velocity.
We will see this later in Fluid mechanics

Reaction time

Reaction time is the time interval between reception of a specific stimulus (observation) and reaction of a person to the stimulus. This time gap occurs due to biological factors and transmission of electrical signals between body organs to brain.

Let your co-experimenter hold the ruler vertically by the top end, so that the zero is at the bottom. Place your hand flat on the table and rest your thumb and forefinger around the 0 cm mark while ensuring that your fingers do not touch the ruler. Without any warning, your partner will drop the ruler. Catch the ruler as quickly as possible by pinching it between your thumb and forefinger. Note the centimeter mark where you caught the ruler. Let this be denoted as h.

Reaction time is obtained using

$$t = \sqrt{\frac{2h}{g}}$$

Sports !

The extreme demand for quick reflexes

- When facing a fast bowler sprinting in at 145 kmph, the ball takes just 0.45 seconds to reach the crease, leaving the batter only about **0.25 seconds** to decide whether to duck or hit a pull shot.
- Legendary Indian wicketkeeper M S Dhoni holds world records for stumping executed in just **0.12 seconds**, which is actually faster than the blink of a human eye.
- At his physical peak, Mike Tyson's combination speed was clocked at approximately **0.15 seconds** from the moment his brain decided to throw a punch to the moment it landed.
- F1 racers have an average reaction time of **0.20 seconds (200 milliseconds)** to visual stimuli.